Natural Modes of Modified Structures

John F. Baldwin and Stanley G. Hutton
The University of British Columbia, Vancouver, Canada

Nomenclature

C = matrix of admixture coefficients for mode shape changes
I = identity matrix
K = symmetric stiffness matrix
$\underline{\ell}$ = modification vector
M = symmetric mass matrix
$\underline{u}$ = modification vector in modal coordinates
x = displacement vector
z = displacement vector in modal coordinates
α = magnitude of local stiffness modification
Δ = change between baseline and modified structures
λ = natural frequency of modified structure
Φ = modal matrix with columns ϕ_k
ϕ_k = kth mode shape vector
ϕ_k' = kth mode shape vector of modified structure
Ω = frequency matrix with diagonals ω^2
ω = natural frequency of baseline structure

Note: Capitalization indicates a matrix.

I. Introduction

THE ever-increasing need to improve the efficiency of structural systems places a growing importance on dynamic analysis in the design or development cycle. A great deal of effort has been directed at defining potential vibration problems and sophisticated testing and analysis procedures now exist to help identify such problems. In particular, the finite element method can be used to obtain a detailed analytical estimate of system characteristics. Thus, from a knowledge of the service and load conditions, potential vibration problems can be identified. At this stage of the design cycle, the proposed structure undergoes a series of modifications aimed at eliminating such problems and, in some way, optimizing the design of the structure. Each step in this process requires the analysis of a modified structure that is often only slightly different from a structure previously analyzed. This complete reanalysis of the structure (or substructure) may be an expensive and time-consuming task and make the detailed refinement of the proposed structure difficult.

It is within this context that the methods of structural dynamics modification have evolved. The dynamic effect of a modification on a structure is treated directly as an analysis problem involving the known dynamic properties of the unmodified structure rather than a complete reanalysis of the modified structure. These methods offer a more efficient approach to the problem of determining the "optimum" modification required to ensure that the structure has satisfactory vibration characteristics.

This paper presents a detailed review of structural dynamics modification techniques. The available literature is sizeable and somewhat disparate, and it is felt that researchers and designers working in this area will benefit from this survey. Furthermore, a number of different approaches to the problem, which at first sight appear unrelated, have been discussed in the literature. In this work an attempt has been made to present these approaches from a unified viewpoint.

Two earlier survey papers document much of the work in this area. Pierson[1] describes many of the sensitivity-based approaches but concentrates mostly on optimization techniques, and Arora[2] provides an excellent review of reanalysis techniques for both the static and dynamic problems. More recently, however, renewed interest in these problems has led to further work in this area and a refinement of available techniques. The following two sections present a description of these developments and their various applications.

II. Modification Techniques

The general problem of structural dynamics modification can be derived by starting with the vibration characteristics of an initial, or baseline, design which are obtained from the solution of the discrete structural vibration problem (typically formulated using finite elements):

$$(K-\omega^2 M)x=0 \tag{1}$$

Solution of this equation yields the eigenvalue matrix Ω with diagonals ω_i^2, and the modal matrix Φ with columns ϕ_i.

Introducing completely arbitrary modifications to the baseline structure stiffness and mass matrices, ΔK and ΔM, the vibration characteristics of the modified structure are obtained in general from the solution of

$$(K-\lambda^2 M-\lambda^2 \Delta M+\Delta K)x=0 \tag{2}$$

where λ^2 is the new eigenvalue of the modified structure. Note that ΔK and ΔM are the stiffness and mass matrices of the actual physical modification and will, in practice, contain many zero elements. Because the configuration of the struc-

Stanley G. Hutton is an Associate Professor of Mechanical Engineering at the University of British Columbia, with research interests in the measurement and analysis of structural dynamics problems. Dr. Hutton received a Bachelor of Science degree from the University of Nottingham, England, and has graduate degrees from the University of Calgary and the University of British Columbia. Prior to his present position he was a senior Lecturer at the University of Adelaide in Australia. Current research activities involve the measurement and analysis of the dynamic response of ship structures, wood machining equipment, and hand-held vibrating tools.

John F. Baldwin is currently a Research Engineer with the Department of Mechanical Engineering at the University of British Columbia. He holds B.A.Sc. and M.A.Sc. degrees in civil engineering from the University of British Columbia. Previous research has been in the areas of finite element applications to fluid mechanics and structural mechanics problems. Current research work involves the analysis of modified ship structures.

Received May 31, 1984; revision received Nov. 15, 1984.

ture is not changed, the mode shapes (eigenvectors) of the baseline structure form a complete vector basis for the modified structure. Hence, transforming to modal coordinates z and premultiplying by Φ^T, Eq. (2) becomes

$$(\Phi^T K\Phi - \lambda^2\Phi^T M\Phi - \lambda^2\Phi^T \Delta M\Phi + \Phi^T \Delta K\Phi)z = 0 \tag{3}$$

For orthonormal modes $\Phi^T K\Phi = \Omega$ and $\Phi^T M\Phi = I$ and, therefore, Eq. (3) becomes

$$(\Omega - \lambda^2 I + \Phi^T \Delta K\Phi - \lambda^2\Phi^T \Delta M\Phi)z = 0 \tag{4}$$

Equation (4) represents the general structural dynamics modification problem and is a new eigenvalue problem for the modified structure with eigenvalues λ_i^2 and mode shapes (in modal coordinates) z_i. Ω and I are diagonal, but for an arbitrary modification ΔK, ΔM, $\Phi^T\Delta K\Phi$, and $\Phi^T\Delta M\Phi$ will be full matrices. In general, the solution of Eq. (4) offers no advantage over a complete reanalysis. However if certain assumptions are made about the form of the modification that is permitted, then it can be shown that significant simplification of Eq. (4) is possible. The nature of the modification allowed determines the type of the resulting modification problem. These alternate techniques of structural dynamics modification can be grouped into three categories: techniques based on small modifications, techniques based on localized modifications, and techniques based on modal approximation. These three basically different formulations of the structural dynamics modification problem will be discussed in some detail.

Techniques Based on Small Modifications

The formulation of a structural dynamics modification technique based on small modifications has been derived by three separate approaches. These are based on Rayleigh's principle, eigenvalue derivatives, and modal perturbation, and lead to very similar formulations. These three approaches to this common formulation illuminate different aspects of the problem and are each considered.

Rayleigh's Method

Perhaps the simplest approach to the problem is to use Rayleigh's method. It is assumed that, for small modifications, the mode shapes do not change appreciably and, hence, can be used in Rayleigh's formula to determine the natural frequencies of the modified structure. The approach is discussed in a finite element text by Irons and Ahmad.[3]

Applying Rayleigh's method, for a baseline structure with a natural frequency ω_i^2 and associated mode shape ϕ_i, to the modified structure gives

$$\lambda_i^2 = (\omega_i + \Delta\omega_i)^2 = \frac{\phi_i^T(K+\Delta K)\phi_i}{\phi_i^T(M+\Delta M)\phi_i} \tag{5}$$

Simplifying and assuming $\Delta\omega_i/\omega_i$ to be small leaves

$$\frac{\Delta\omega_i}{\omega_i} = \frac{\phi_i^T\Delta K\phi_i - \omega_i^2\phi_i^T\Delta M\phi_i}{2\phi_i^T K\phi_i} \tag{6}$$

This is an expression for the relative change in a natural frequency due to a structural modification that only involves the vibration characteristics of the baseline structure and the structural modifications. Equation (6) shows that the mode shape of the baseline structure provides sensitivities for simple point mass or stiffness modifications.

Sensitivity Approach

A more rigorous approach is to consider the structural modification problem in terms of a rate of change of an eigenvalue with respect to a structural parameter change.

Consider the derivative of Eq. (1) with respect to a change in the jth structural parameter,

$$\left(\frac{\partial K}{\partial p_j} - \frac{\partial\omega_i^2}{\partial p_j}M - \omega_i^2\frac{\partial M}{\partial p_j}\right)\phi_i + (K - \omega_i^2 M)\frac{\partial\phi_i}{\partial p_j} = 0 \tag{7}$$

Premultiplying by ϕ_i^T and solving for $\partial\omega_i^2/\partial p_j$ [noting that the last term drops out because of Eq. (1) and symmetry of K and M] gives

$$\frac{\partial\omega_i^2}{\partial p_j} = \left[\phi_i^T\left(\frac{\partial K}{\partial p_j} - \omega_i^2\frac{\partial M}{\partial p_j}\right)\phi_i\right]\Big/\phi_i^T M\phi_i \tag{8}$$

This may then be used in a Taylor series expansion to give a first-order approximation to the natural frequencies of the modified structure. Hence,

$$\omega_i^2(p_j) = \omega_i^2(p_{j0}) + (p_j - p_{j0})\frac{\partial\omega_i^2}{\partial p_j} \tag{9}$$

Then substituting Eq. (8) into Eq. (9) and assuming unit modal mass,

$$\Delta\omega_i^2 = \Delta p_j\phi_i^T\left(\frac{\partial K}{\partial p_j} - \omega_i^2\frac{\partial M}{\partial p_j}\right)\phi_i \tag{10}$$

Equation (10) provides an expression for the effect of a structural modification on the natural frequencies of the system. For a small parameter change $\Delta p_j(\partial K/\partial p_j)$ is just ΔK, and Eq. (10) reduces to the same form as Eq. (6). Equation (9) shows that, to a first-order approximation, the change in frequency does not depend on the change in mode shape. It is this fact that allows the modification methods to offer an efficient alternative to a complete reanalysis for small-magnitude structural modifications. Early work in this area was conducted by McCalley[4] and Wittrick.[5] Farshad[6] has considered the related continuum problem and, in general, the mathematical foundations have been discussed by Lancaster.[7]

A major treatment of the entire problem, including the calculation of the modified structure mode shapes using the derivative approach, was presented by Fox and Kapoor.[8] The first-order solution was discussed. Two methods were derived to calculate the mode shape derivatives. The first method expressed the eigenvector derivatives in terms of a series expansion in the unmodified eigenvectors and, hence, required the knowledge of the full modal characteristics of the baseline structure (although truncation is possible). A second method, which is potentially more attractive, was also derived. It expressed the eigenvector derivative only in terms of the corresponding frequency and eigenvector of the baseline structure. However, numerical difficulties were encountered with this second method and prevented its successful implementation. Kiefling[9] in a short Note suggested possible improvements to overcome these numerical difficulties. These were incorporated by Hasselman and Hart[10] but were not found to alleviate the problem. The difficulty was eventually solved by Nelson,[11] whose method also took advantage of the banded nature of the equations. Whitesell[12] presented a method of calculating the eigenvector derivatives that uses only $\mathcal{O}(n^2)$ calculations (where n = size of system) independent of the sparsity of the equations. [The eigenvalue problem in general uses $\mathcal{O}(n^3)$ calculations.]

The classical methods of sensitivity analysis assume that the structure is linear and kinematically determinate. This simplifies the sensitivity calculations because the compatibility conditions are not influenced by the structural parameters. For the displacement formulation of the finite element method, the compatibility conditions represent the element connectivities only and are independent of the structural parameters. Using analogies to electrical network theory, Van Belle[13] presents a theory of adjoint structures to calculate the differential sensitivities of mechanical structures. A flexibility approach is used. This work has been extended by Vanhonacker[14] to calculate the sensitivities of natural frequencies and mode shapes. The effects of viscous damping have been included.

Though not adding to the classical structural vibration problem, the calculation of derivatives of eigenvalues and eigenvectors has been extended to include unsymmetric and non-self-adjoint systems.[11,15-18] Taylor and Kane[19] discuss the quadratic eigenvalue problem. These methods are a formal generalization of Fox and Kapoor's work[8] and could be of interest in problems with nonclassical damping.

The differential sensitivity methods have also been extended to handle the finite modification problem. Second-order terms have been included in a Taylor series expansion of the eigenvalue problem about the unmodified structure. Miura and Schmit,[20] Vanhonacker,[14] Van Belle,[21] Rizai and Bernard,[22] and Rudisill and Bhatia[23] have presented formulations of this.

Perturbation Approaches

A third approach which also leads to sensitivity-type formulations involves the use of perturbation methods and is essentially a generalization of Rayleigh's method. These methods provide a clearer understanding of the involvement of the mode shapes. Equation (1) can be rewritten in the form

$$\Phi^T K \Phi = \Phi^T M \Phi \Omega \tag{11}$$

If changes are introduced to the system as

$$K' = K + \Delta K, \qquad M' = M + \Delta M$$

$$\Omega' = \Omega + \Delta\Omega, \qquad \Phi' = \Phi + \Delta\Phi \tag{12}$$

then Eq. (11) becomes, for the modified structure,

$$(\Phi+\Delta\Phi)^T (K+\Delta K)(\Phi+\Delta\Phi)$$

$$= (\Phi+\Delta\Phi)^T (M+\Delta M)(\Phi+\Delta\Phi)(\Omega+\Delta\Omega) \tag{13}$$

It is convenient to express the new mode shapes as

$$\Phi' = \Phi + \Delta\Phi = \Phi (I+C)^T \tag{14}$$

where C has zero diagonal elements and defines the contribution of original modes to the modified modes. Neglecting terms of Δ^2 and higher, and assuming that the mode shapes have been normalized to have unit modal mass, Eq. (13) becomes

$$\Phi^T \Delta K \Phi - \Omega \Phi^T \Delta M \Phi = \Delta\Omega + C^T \Omega - \Omega C^T \tag{15}$$

Equation (15) is a matrix equation involving structural changes on the left, whereas on the right $\Delta\Omega$ is a diagonal matrix of frequency changes, and the terms involving C are all off-diagonal and represent the participation of other baseline mode shapes in a mode shape change. Notice that the diagonal terms are of the same form as Eq. (10).

The first treatment of the structural modification problem was by Rayleigh,[24] who used this type of approach and derived an approximate solution in terms of modal coordinates from energy expressions. Jones[25] extended this work to include general perturbations and derived expressions for frequency and mode shape changes. The derivation presented was confined to systems with no inertial coupling, but it was the first to use the fact that the new mode shapes may be expressed in terms of a series of the unmodified mode shapes, since these form a set of basis functions for both systems. The perturbation problem was also presented in a more general mathematical form by Wilkinson[26] and Bellman.[27]

Romstad et al.[28] investigated a variety of more general perturbation formulations using a power series approach. Finite perturbations were considered and several second-order approximations were investigated for a buckling eigenvalue problem. White and Mayton[29] also examined the perturbation problem. The earlier work by Jones[25] also considered finite perturbations, however, because of computation barriers, only simple infinitesimal examples were considered.

Stetson[30,31] presented a powerful application by using holographic interferometry to experimentally determine the vibration modes for a prototype and then predicted the effect of small changes to the prototype analytically by means of the perturbation equations. The perturbation equations derived are similar to Jones[25] but allow for inertial coupling.

Of considerable practical interest is the solution to the inverse problem of determining the structural modifications necessary to meet specified constraints on frequencies and mode shapes. Stetson et al.[32-34] have extended the modal perturbation approach to treat this problem. Theirs was one of the first formulations of the reanalysis problem that considered constraints on both the frequency and the mode shapes. Sandstrom and Anderson[35] extended this work by relating the physical changes in frequency and mode shape directly to changes in structural parameters. This is an important difficulty for practical applications. These algorithms have been implemented into a general-purpose program.[36]

Sandstrom, Anderson, and others at the University of Michigan[37-39] report that the linear energy formulation [Eq. (15)] gives good accuracy for frequency goals, but is often not accurate for significant mode shape changes.[39] They have further generalized the capabilities of the modal perturbation formulations by extending the treatment to include finite modifications. They developed two competing methods, one using nonlinear mathematical programming[39] and the other involving an incremental predictor-corrector approach.[38] The methods have been successfully applied to large structural redesign problems.

Techniques Based on Localized Modifications

A completely different formulation of the structural modification problem has been derived by considering modifications of a localized nature but of arbitrary magnitude. In 1948, Young[40] presented an analytical method for a beam modification problem that used the modal properties of the beam in a characteristic equation whose roots were the eigenvalues of the modified system. Lee and Saibel[41] used a slightly different approach to derive a characteristic determinant having the same order as the number of stiffness modifications made in the system. Das and Navaratna[42] extended Young's work to treat two-dimensional systems and analyzed a rectangular-plate problem. None of these early references was concerned with finding the mode shapes of the modified structure.

Using similar techniques, Weissenburger[43,44] developed the Eigenvalue Modification Method to give the complete solution to the localized structural modification problem including the new mode shapes. This technique can be discussed by considering the general structural modification problem as described by Eq. (4). For simplicity, consider only stiffness modifications and let the stiffness modification be localized so that it may be expressed as a vector product

$$\Delta K = \alpha \underline{\ell}\,\underline{\ell}^T \tag{16}$$

where α is the stiffness constant and $\underline{\ell}$ a column vector used to define the location of the modification. By defining $\underline{u} = \Phi^T \underline{\ell}$, Eq. (4) may then be written as

$$(\Omega - \lambda^2 I + \alpha \underline{u}\underline{u}^T) z = 0 \tag{17}$$

Equation (17) is thus the eigenvalue problem resulting from a localized stiffness modification. The advantages of the eigenvalue modification method now appear. By considering a general row of the matrix equation (17), the unknown components of the eigenvector z can be expressed directly in terms of the baseline frequencies and can be used to uncouple Eq. (17) to give

$$\sum_k \frac{u_k^2}{(\omega_k^2 - \lambda^2)} = -\frac{1}{\alpha} \tag{18}$$

Therefore, the eigenvalues of the modified structure λ^2 are just the roots of Eq. (18). A Newton-Raphson method is used to solve this characteristic equation. Notice that unlike the perturbation or sensitivity methods, the eigenvalue modification formulation is exact for arbitrarily large (albeit local) modifications.

Pomazal[45] and Pomazal and Snyder[46] have generalized Weissenburger's work to incorporate nonclassical damping, unsymmetrical matrices, and multiple eigenvalues. Also, O'Callahan and Avitabile[47] have considered systems with high damping ratios and complex modes. Complex modes were felt to be important when the modifications tended to increase the critical damping ratio or make the damping nonproportional.[47]

A method known as Diakoptics was developed by Kron[48] for the piecewise solution of large-scale structures. This procedure has been extended by Simpson and Tabarok[49] and Simpson.[50,51] Hallquist[52,53] and Hallquist and Snyder[54-57] have incorporated these concepts into the eigenvalue modification method to generalize its formulation of the structural modification problem. Stiffness, mass, and damping modifications can be made with equal ease. Successive modifications are discussed and it is shown that the connection of substructures can be treated as a structural modification problem. The effects of modal truncation as an approximation are also examined.

Hallquist also discussed a method of considering nonlocal modifications. The eigenvalue modification method is based on the expression of a local structural modification as a so-called tie vector. Hallquist[52] showed that a nonlocal modification could be expressed as a series of tie vectors by a spectral decomposition of the modification. The eigenvectors of this decomposition then represented a series of such vectors that could be used to modify the structure successively. A direct solution with a set of tie vectors was also derived which yielded a characteristic determinant having the same order as the number of ties. Hirai et al.[58] also proposed a similar approach. Wang et al.[59-61] and Palazzolo[62] have extended this approach and implemented it in a general modification program.[60]

The spectral decomposition of a general modification into a series of tie vectors has been examined further by Chou[63] and O'Callahan and Chou,[64] who have implemented a general modification procedure for a three-dimensional beam element. Luk[65] and Luk and Mitchell[66-68] have also derived a similar treatment for three-dimensional truss elements.

Techniques Based on Modal Approximation

A third formulation of the modification problem, utilizing a condensation-type method, has been used by several researchers.[28,69] The foundation for all of the structural modification methods was the fact that the eigenvectors of the unmodified structure form a complete vector basis to describe the motion of any modified structure. The strength of these modification procedures came from the fact that the modifications could be handled more easily when expressed in modal coordinates. In practice, only a partial modal set is generally known, hence the transformation to modal coordinates also represents a projection onto a modal subspace. The full modification eigenvalue problem of order N is projected onto a partial modal space of order M (which is much less than N). This condensed eigenvalue problem can be solved directly since M will typically be an order of magnitude less than N. Note that this approach does not assume that the modifications are either small or localized. Instead it represents an approximate solution to the general modification eigenvalue problem equation (4). Notice also that, in practice, the two alternate formulations based on restrictions to finite or localized modifications must also be solved with a truncated modal set. This raises important considerations about the convergence capabilities of all three formulations. In practice this does not appear to be a problem, but few general conclusions have been reported for any of the formulations. Romstad et al.[28] have presented work in this area.

III. Applications

The methods of structural dynamics modification have seen application to a variety of problems. Foremost is the structural redesign problem where the savings offered by an efficient solution to the modified eigenvalue problem are very important.[69,36,60] This reanalysis is also an integral part of explicit optimization schemes and the methods described here (especially the sensitivity approach) have been of great relevance to computer optimization techniques,[1,70-77] many of which are concerned with minimum weight design. A related problem has been the statistical variation of structural parameters and its effects on the dynamics of structures.[78,79,4]

In recent years these methods have begun to offer a very powerful bridge between experimental and analytical work. The problem of correlating an analytical model with experimentally obtained dynamic results was treated using a sensitivity approach by Taylor.[80] Hypothetical analytical modifications to an experimental model have also been considered.[30,81] The expanding minicomputer technology has helped to create powerful tools for dynamic troubleshooting by combining modal analysis and dynamic testing[82] with the eigenvalue modification method.[83-85] Recent developments in this area were reported at the Second International Modal Analysis Conference.[60,64,67-69,86,87]

Although the developments described herein have been in terms of the structural vibration problem, the methods presented are of course applicable to the general eigenvalue problem. Other eigenvalue problems that have been considered are the flutter problem[23] and the buckling problem.[5,28]

Structural dynamics modification techniques have been incorporated into a variety of commercially available programs. These cover a wide range of capabilities and are based on one or more of the three formulations described in Sec. II. For the most part, the programs fall into two categories. One type is primarily oriented toward minicomputers and often ties into experimental modal analysis programs. The other type is oriented toward redesign of large structures and, hence, run on mainframe computers and generally interface with a general-purpose finite element program such as NASTRAN. Examples of the first type are marketed by Structural Measurement Systems, ENTEK, ZONIC, NICOLET, Structural/Kinematics, Structural Dynamics Research Corp., Hewlett Packard, and Ref. 69, while examples of the second type are presented in Refs. 60 and 36.

IV. Conclusions

A review of the current state-of-the-art in predicting the effects of structural modifications on the dynamic response of structures has been presented. Analytical formulations of the structural dynamics modification problem fall into three main categories based on differing assumptions about the form of the structural modification. It is these assumptions about the nature of the modifications that allow for a direct solution to the modified structure using the known dynamic characteristics of the original structure, hence avoiding a complete reanalysis. The eigenvalue modification method is a technique based on localized modifications that leads to an exact solution of the problem. The modal perturbation and sensitivity methods are based on the assumption of small modifications, and the modal condensation method is an approximate technique based on modal truncation.

These techniques appear to be well developed and offer differing advantages. The eigenvalue modification method is best suited to handle the exact solution of arbitrarily large, simple spring and mass modifications. Extensive modifications and direct changes to structural elements are difficult to handle with this procedure. The modal perturbation and sensitivity methods are best suited to finite modifications of general structural parameters. The methods give linear formulations for changes on the order of 5%, and by including higher order nonlinear terms can be used for changes on the order of 50%. The modal perturbation method has also been used to formulate the inverse problem of determining the structural modifications necessary to meet specified constraints on natural frequencies and mode shapes, including optimization criteria. The modal condensation method offers a general formulation in the form of a reduced eigenvalue problem. This method appears to be quite attractive for finite changes on the order of 50%.

The eigenvalue modification method has typically been applied to mechanical engineering fields, and programs have been developed to run on minicomputers and interface with experimental modal analysis packages. The aim of this work has been to help with the optimization of prototype development of mechanical structures. The modal perturbation and modal condensation methods, on the other hand, have typically seen application to civil engineering problems. Programs have been developed to run on mainframe computers and interface with general-purpose finite element programs. The aim of this work has been to further the optimization process in large-scale structural design, particularly where dynamic constraints and minimum weight are important.

Critical areas of modification theory not fully covered in the literature include the effects of modal truncation on the convergence characteristics of all three techniques. Further, although the techniques are moderately well developed, the relative performance of the various methods is only poorly defined for different classes of problems.

Acknowledgment

This work was performed under Contract 8SC82-00327, "Analysis and Measurement of the Effect of Structural Modification to Ship Structures," from the Department of National Defence, Defense Research Establishment Atlantic, Halifax, Canada.

References

[1]Pierson, B. L., "A Survey of Optimal Structural Design Under Dynamic Constraints," *International Journal for Numerical Methods in Engineering*, Vol. 4, July-Aug. 1972, pp. 491-499.

[2]Arora, J. S., "Survey of Structural Reanalysis Techniques," *Journal of the Structural Division, American Society of Civil Engineers*, Vol. 102, No. ST4, April 1976, pp. 783-803.

[3]Irons, B. and Ahmad, S., *Techniques of Finite Elements*, Ellis Horwood Ltd., Chichester, England, 1980, pp. 360-361.

[4]McCalley, R. B. Jr., "Error Analysis for Eigenvalue Problems," *Conference Papers—American Society of Civil Engineers, 2nd Conference on Electronic Computation*, ASCE, New York, 1960, pp. 523-550.

[5]Wittrick, W. H., "Rates of Change of Eigenvalues, with Reference to Buckling and Vibration Problems," *Journal of the Royal Aeronautical Society*, Vol. 66, Sept. 1962, pp. 590-591.

[6]Farshad, M., "Variations of Eigenvalues and Eigenfunctions in Continuum Mechanics," *AIAA Journal*, Vol. 12, April 1974, pp. 560-561.

[7]Lancaster, P., "On Eigenvalues of Matrices Dependent on a Parameter," *Numerische Mathematik*, Vol. 6, No. 5, Dec. 1964, pp. 377-387.

[8]Fox, R. L. and Kapoor, M. P., "Rates of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, Dec. 1968, pp. 2426-2429.

[9]Kiefling, L. A., "Comment on 'the Eigenvalue Problem for Structural Systems with Statistical Properties'," *AIAA Journal*, Vol. 8, July 1970, p. 1371.

[10]Hasselman, T. K. and Hart, G. C., "Modal Analysis of Random Structural Systems," *Journal of the Engineering Mechanics Division, American Society of Civil Engineers*, June 1972, pp. 561-579.

[11]Nelson, R. B., "Simplified Calculation of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, Sept. 1976, pp. 1201-1205.

[12]Whitesell, J. E., "Design Sensitivity in Dynamical Systems," Ph.D. Dissertation, Dept. of Mechanical Engineering, Michigan State University, Ann Arbor, MI, 1980.

[13]Van Belle, H., "Theory of Adjoint Structures," *AIAA Journal*, Vol. 14, July 1976, pp. 977-979.

[14]Vanhonacker, P., "Differential and Difference Sensitivities of Natural Frequencies and Mode Shapes of Mechanical Structures," *AIAA Journal*, Vol. 18, Dec. 1980, pp. 1511-1514.

[15]Rogers, L. C., "Derivatives of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 8, May 1970, pp. 943-944.

[16]Rudisill, C. S., "Derivatives of Eigenvalues and Eigenvectors for a General Matrix," *AIAA Journal*, Vol. 12, May 1974, pp. 721-722.

[17]Rudisill, C. S. and Chu, Y. Y., "Numerical Method for Evaluating the Derivatives of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 13, June 1975, pp. 834-836.

[18]Plaut, R. H. and Huseyin, K., "Derivatives of Eigenvalues and Eigenvectors in Non-Self Adjoint Systems," *AIAA Journal*, Vol. 11, Feb. 1973, pp. 250-251.

[19]Taylor, D. L. and Kane, T. R., "Multiparameter Quadratic Eigenvalue Problems," *Journal of Applied Mechanics*, Vol. 42, June 1975, pp. 478-483.

[20]Miura, H. and Schmit, L. A. Jr., "Second Order Approximation of Natural Frequency Constraints in Structural Synthesis," *International Journal for Numerical Methods in Engineering*, Vol. 13, 1978, pp. 337-351.

[21]Van Belle, H., "Higher Order Sensitivities in Structural Systems," *AIAA Journal*, Vol. 20, Feb. 1982, pp. 286-288.

[22]Rizai, N. M. and Bernard, J. E., "An Efficient Method for Predicting the Dynamic Effects of a Redesign," *Proceedings of the 1st International Modal Analysis Conference*, Orlando, FL, Nov. 1982, pp. 436-442.

[23]Rudisill, C. S. and Bhatia, K. G., "Second Derivatives of the Flutter Velocity and the Optimization of Aircraft Structures," *AIAA Journal*, Vol. 12, May 1974, pp. 721-722.

[24]Rayleigh, B., *The Theory of Sound*, Vol. I, Dover, New York, 1945, pp. 109-118, 214-217.

[25]Jones, R. P. N., "The Effect of Small Changes in Mass and Stiffness on the Natural Frequencies and Modes of Vibrating Systems," *International Journal of Mechanical Sciences*, Vol. 1, No. 4, July 1960, pp. 350-355.

[26]Wilkenson, J. H., *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965 (ref. p. 67).

[27]Bellman, R., *Introduction to Matrix Analysis*, 2nd Ed., McGraw-Hill Book Co., New York, 1970, pp. 60-63.

[28]Romstad, K. M., Hutchinson, J. R., and Runge, K. H., "Design Parameter Variation and Structural Response," *International Journal for Numerical Methods in Engineering*, Vol. 5, Jan.-Feb. 1973, pp. 337-349.

[29]White, C. W. and Mayton, B. D., "Eigensolution Sensitivity to Parametric Modal Perturbations," *Shock and Vibration Bulletin*, Vol. 46, No. 5, 1976, pp. 123-133.

[30]Stetson, K. A., "Perturbation Method of Structural Design Relevant to Holographic Vibration Analysis," *AIAA Journal*, Vol. 13, April 1975, pp. 457-459.

[31]Flax, A. H., "Comment on 'Perturbation Methods of Structural Design Relevant to Holographic Vibration Analysis'," *AIAA*

Journal, Vol. 14, March 1976, p. 416.

[32]Stetson, K. A. and Palma, G. E., "Inversion of First Order Perturbation Theory and Its Application to Structural Design," *AIAA Journal*, Vol. 14, April. 1976, pp. 454-460.

[33]Stetson, K. A., Harrison, I. R., and Palma, G. E., "Redesigning Structural Vibration Modes by Inverse Perturbation Subject to Minimal Change Theory," *Computer Methods in Applied Mechanics and Engineering*, Vol. 16, 1978, pp. 151-175.

[34]Stetson, K. A. and Harrison, I. R., "Redesign of Structural Vibration Modes by Finite Element Inverse Perturbations," *Transactions of ASME, Journal of Engineering for Power*, Vol. 103, April 1981, pp. 319-325.

[35]Sandstrom, R. E. and Anderson, W. J., "Modal Perturbation Methods for Marine Structures," *Transactions of the Society of Naval Architects and Marine Engineers*, Vol. 90, 1982, pp. 41-54.

[36]Hoff, C. J. and Anderson, W. J., "Linear Dynamic Redesign," *University of Michigan LDRUM Version 1.0 Theoretical and User's Manual*, Dept. of Aerospace Engineering, University of Michigan, Ann Arbor, MI, May 31, 1983.

[37]Bernitsas, M. M., Hoff, C. J., and Kokarakis, J. E., "Nonlinear Inverse Perturbation in Structural Dynamics Redesign of Risers," Preprint, Dept. of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor, MI, submitted to *Transactions of ASME, Journal of Energy Resources Technology*.

[38]Hoff, C. J., Bernitsas, M. M., Sandstrom, R. E., and Anderson, W. J., "Inverse Perturbation Method for Structural Redesign with Frequency and Mode Shape Constraints," *AIAA Journal*, Vol. 22, Sept. 1984, pp. 1304-1309.

[39]Kim, K. O., Anderson, W. J., and Sandstrom, R. E., "Nonlinear Inverse Perturbation Method in Dynamic Analysis," *AIAA Journal*, Vol. 21, Sept. 1983, pp. 1310-1316.

[40]Young, D., "Vibration of a Beam with Concentrated Mass, Spring, and Dashpot," *Journal of Applied Mechanics*, Vol. 15, 1948, pp. 65-72.

[41]Lee, W. F. Z. and Saibel, E., "Free Vibrations of Constrained Beams," *Journal of Applied Mechanics*, Vol. 19, Dec. 1952, pp. 471-477.

[42]Das, Y. C. and Navaratna, D. R., "Vibrations of a Rectangular Plate with Concentrated Mass, Spring and Dashpot," *Journal of Applied Mechanics*, Vol. 30, March 1963, pp. 31-36.

[43]Weissenburger, J. T., "Effect of Local Modifications on the Vibration Characteristics of Linear Systems," *Journal of Applied Mechanics*, Vol. 35, June 1968, pp. 327-332.

[44]Weissenburger, J. T., "The Effect of Local Modifications on the Eigenvalues and Eigenvectors of Linear Systems," D.Sc., Sever Institute, Washington University, St. Louis, MO, Jan. 1966.

[45]Pomazal, R., "The Effect of Local Modification on the Eigenvalues and Eigenvectors of Damped Linear Systems," Ph.D. Dissertation, Michigan Technical University, Houghton, MI, 1969.

[46]Pomazal, R. J. and Snyder, V. W., "Local Modifications of Damped Linear Systems," *AIAA Journal*, Vol. 9, Nov. 1971, pp. 2216-2221.

[47]O'Callahan, J. C. and Avitabile, P., "A Structural Modification Procedure Using Complex Modes," *Proceedings of the 1st International Modal Analysis Conference*, Orlando, FL, Nov. 1982, pp. 418-422.

[48]Kron, G., *Diakoptics: The Piecewise Solution of Large Scale Systems*, MacDonald & Co. Ltd., London, England, 1963.

[49]Simpson, A. and Tabarook, B., "On Kron's Eigenvalue Procedure and Related Methods of Frequency Analysis," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 21, Pt. 1, 1968, pp. 1-39.

[50]Simpson, A., "A Generalization of Kron's Eigenvalue Procedure," *Journal of Sound and Vibration*, Vol. 26, Jan. 1973, pp. 129-139.

[51]Simpson, A., "Kron's Method: A Consequence of the Minimization of the Primitive Lagrangian in the Presence of Displacement Constraints," *Journal of Sound and Vibration*, Vol. 27, March 1973, pp. 377-386.

[52]Hallquist, J., "Modification and Synthesis of Large Dynamic Structural Systems," Ph.D. Dissertation, Michigan Technical University, Houghton, MI, 1974.

[53]Hallquist, J., "An Efficient Method for Determining the Effects of Mass Modifications in Damped Systems," *Journal of Sound and Vibration*, Vol. 44, No. 3, 1976, pp. 449-459.

[54]Hallquist, J. and Snyder, V. W., "Synthesis of Two Discrete Vibratory Systems Using Eigenvalue Modification," *AIAA Journal*, Vol. 11, Feb. 1973, pp. 247-249.

[55]Halliquist, J. O. and Snyder, V. W., "Linear Damped Vibratory Structures with Arbitrary Support Conditions," *Journal of Applied Mechanics*, Vol. 40, No. 3, 1973, pp. 312-313.

[56]Hallquist, J. O. and Snyder, V. W., "An Exact Piecewise Analysis of Large Viscously Damped Structural Systems," *Developments in Mechanics*, Vol. 7, *Proceedings of the 13th Midwestern Mechanics Conference*, School of Engineering, University of Pittsburgh, Aug. 1973, pp. 457-472.

[57]Hallquist, J. O. and Snyder, V. W., "On the Connection of Viscously Damped Continuous Vibratory Systems," *Journal of Sound and Vibration*, Vol. 32, No. 1, 1974, pp. 131-142.

[58]Hirai, I., Yoshimura, T., and Takamura, K., "Short Communication on a Direct Eigenvalue Analysis for Locally Modified Structures," *International Journal for Numerical Methods in Engineering*, Vol. 6, 1973, pp. 441-456.

[59]Wang, B. P. and Chu, F. H., "Effective Dynamic Reanalysis of Large Structures," *Shock and Vibration Bulletin*, Vol. 51, No. 3, 1981.

[60]Wang, B. P., Chu, F. H., and Gross, D. W., "REANALYSIS—A User Oriented Computer Program for the Reanalysis of Structural Systems," *Proceedings of the 2nd International Modal Analysis Conference*, Union College, Schenectady, NY, Feb. 1984, pp. 940-944.

[61]Wang, B. P., Palazzolo, A. B., and Pilkey, W. D., "Reanalysis, Modal Synthesis, and Dynamic Design," *State of the Art Survey of Finite Element Methods*, ASME, New York, 1982, Chap. 8.

[62]Palazzolo, A. B., "Vibration of Locally Modified Mechanical and Structural Systems," Ph.D. Dissertation, University of Virginia, Charlottesville, VA, 1981.

[63]Chou, C. M., "Structural Dynamics Modification of 3D Beam Elements Using a Local Eigenvalue Modification Procedure," Ph.D. Dissertation, University of Lowell, Lowell, MA, 1984.

[64]O'Callahan, J. C. and Chou, C. M., "Study of a Structural Modification Procedure with Three Dimensional Beam Elements Using a Local Eigenvalue Modification Procedure," *Proceedings of the 2nd International Modal Analysis Conference*, Union College, Schenectady, NY, Feb. 1984, pp. 945-952.

[65]Luk, Y. W., "System Modeling and Modification via Modal Analysis," Ph.D. Dissertation, Dept. of Mechanical Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA, Aug. 1981.

[66]Luk, Y. W. and Mitchell, L. D., "System Modeling and Modification via Modal Analysis," *Proceedings of the 1st International Modal Analysis Conference*, Union College, Schenectady, NY, Nov. 1982, pp. 423-429.

[67]Luk, Y. W. and Mitchell, L. D., "Analytical and Geometric Substructuring via Dual Modal Space Structural Modification," *Proceedings of the 2nd International Modal Analysis Conference*, Union College, Schenectady, NY, Feb. 1984, pp. 50-57.

[68]Luk, Y. W. and Mitchell, L. D., "Implementation of the Dual Modal Space Structural Modification Method," *Proceedings of the 2nd International Modal Analysis Conference*, Union College, Schenectady, NY, Feb. 1984, pp. 930-936.

[69]Djordjevic, W., Tseng, T-M., and Anagnostis, S., "EDASP: Structural Modification Program," *Proceedings of the 2nd International Modal Analysis Conference*, Union College, Schenectady, NY, Feb. 1984, pp. 75-81.

[70]Bellagamba, L. and Yang, T. Y., "Minimum Mass Truss Structure with Constraints on Fundamental Natural Frequency," *AIAA Journal*, Vol. 19, Nov. 1981, pp. 1452-1458.

[71]Fox, R. L. and Kapoor, M. P., "Structural Optimization in the Dynamics Response Regime: A Computational Approach," *AIAA Journal*, Vol. 8, Oct. 1970, pp. 1798-1804.

[72]Romstad, K. M., "Optimal Design for Dynamic Response," *Proceedings of the Symposium on Structural Dynamics*, Vol. 1, Loughborough University of Technology, Loughborough, 1970, pp. A.2.1-A.2.18.

[73]Rubin, C. P., "Minimum Weight Design of Complex Structures Subject to a Frequency Constraint," *AIAA Journal*, Vol. 8, 1970, pp. 923-927.

[74]Zarghamee, M. S., "Optimum Frequency of Structures," *AIAA Journal*, Vol. 6, April 1968, pp. 749-750.

[75]Zarghamee, M. S., "Minimum Weight Design with Stability Constraint," *Journal Structural Division, American Society of Civil Engineers*, Vol. 96, No. ST8, Aug. 1970, pp. 1697-1710.

[76]Olhoff, N., "A Survey of the Optimal Design of Vibrating Structural Elements, Part I: Theory; Part II: Application," *The Shock and Vibration Digest*, Vol. 8, No. 8, 1976, pp. 3-10; No. 9, pp. 3-10.

[77]Kiusalaas, J. and Shaw, R. C., "An Algorithm for Optimal Structural Design with Frequency Constraints," *International Journal for Numerical Methods in Engineering*, Vol. 13, No. 2, 1978, pp.

283-295.

[78]Collins, J., Hart, G., Hasselman, T., and Kennedy, B., "Statistical Identification of Structures," *AIAA Journal*, Vol. 12, Feb. 1974, pp. 185-190.

[79]Collins, J. D. and Thompson, W. T., "The Eigenvalue Problem for Structural Systems with Statistical Properties," *AIAA Journal*, Vol. 7, April 1969, pp. 642-647.

[80]Taylor, J. E., "Scaling a Discrete Structural Model to Match Measured Modal Frequencies," *AIAA Journal*, Vol. 15, Nov. 1977, pp. 1647-1649.

[81]Vanhonacker, P., "Sensitivity Analysis of Mechanical Structures, Based on Experimentally Determined Modal Parameters," *Proceedings of the 1st International Modal Analysis Conference*, Union College, Schenectady, NY, Nov. 1982, pp. 534-541.

[82]Potter, R., "A General Theory of Modal Analysis for Linear Systems," *Shock and Vibration Digest*, Vol. 7, No. 11, Nov. 1975, pp. 3-11.

[83]Herbert, M. R. and Kientzy, D. W., "Applications of Structural Dynamics Modification," SAE Paper 801125, Oct. 1980.

[84]Formenti, D. and Welaratna, S., "Structural Dynamics Modification—An Extension of Modal Analysis," SAE Paper 811043, Oct. 1981.

[85]Ramsey, K. A. and Firmin, A., "Experimental Modal Analysis, Structural Modifications and FEM Analysis—Combining Forces on a Desktop Computer," *Proceedings of the 1st International Modal Analysis Conference*, Union College, Schenectady, NY, Nov. 1982, pp. 443-455.

[86]Mitchell, L. D. and Mitchell, L. D., "Modal Analysis Bibliography—An Update 1980-1983," *Proceedings of the 2nd International Modal Analysis Conference*, Union College, Schenectady, NY, Feb. 1984, pp. 1098-1114.

[87]Crowley, J., Klosterman, A., Rocklin, G., and Vold, H., "Direct Structural Modification Using Frequency Response Functions," *Proceedings of the 2nd International Modal Analysis Conference*, Union College, Schenectady, NY, Feb. 1984, pp. 58-65.